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Stability Bounds for the Control of Large Space Structures

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Introduction

STUDIES of future directions for the space program, such as Ref. 1, have identified a number of potentially important new space initiatives which will require large lightweight space structures with dimensions from one hundred to several thousands of meters. Control systems design for such structures is a challenging problem because the "plant" consists of infinite modes. Of these modes, only a finite number can be modeled and even fewer can be controlled due to practical limitations. Such tools as linear-quadratic Gaussian (LQG) theory[2] can be used for designing reduced-order controllers for such systems. Unfortunately, when going to reduced-order regulators and estimators, stability is no longer assured. The problem of control of large space structures (LSS) using the LQG approach then becomes one of developing new techniques for designing stable, robust, reduced-order regulators and estimators. Balas has discussed the stability problem of reduced-order regulators and estimators in terms of control and observation "spillover" in Ref. 3. The term "control spillover" was used to define that part of the feedback control which excites the uncontrolled (or residual) modes and "observation spillover" was used to define that part of the measurement contaminated by residual modes.

In this paper two different bounds are derived on spectral norms of control and observation spillover terms, any of which, if satisfied, assures asymptotic stability. The bounds are derived using Lyapunov methods. Numerical results are given for a long free-free beam.

System Description and Definitions

The mathematical model of an LSS can be written as

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x_c \\ x_r \end{bmatrix} + \begin{bmatrix} B_c \\ B_r \end{bmatrix} u + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1}$$

where x_c is the n_c-dimensional state vector for the controlled part of the system which consists of the rigid-body modes and some structural modes; x_r is the n_r-dimensional state vector for the uncontrolled (or residual) modes; A_c, A_r, B_c, B_r are appropriately dimensioned system and input matrices; u is the m-dimensional input vector; and $v=(v_1^T, v_2^T)^T$ is the zero-mean Gaussian white noise input vector with covariance intensity matrix V. The l-dimensional observation vector is given by

$$y=C_cx_c+C_rx_r+w \tag{2}$$

where C_c and C_r are $l\times n_c$ and $l\times n_r$ matrices and w is a zero-mean Gaussian white noise. Let the controller be given by

$$u=Gz \tag{3}$$

where G is an $m\times p$ constant matrix, and z is a p-dimensional compensator state vector

$$\dot{z}=A_{cz}z+B_zu+Ky \tag{4}$$

A_{cz}, B_z, and K are appropriately dimensioned constant matrices. For example, an LQG controller designed for the x_c-system (truncating the residual system) has this structure. A regulator-observer system using pole placement would also have the same structure.

The resulting closed-loop equations, ignoring noise terms are

$$\begin{bmatrix} \dot{x}_c \\ \dot{z} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_c & B_cG & 0 \\ KC_c & A_{cz}+B_zG & 0 \\ 0 & 0 & A_r \end{bmatrix} \begin{bmatrix} x_c \\ z \\ x_r \end{bmatrix} + \begin{bmatrix} 0 \\ KC_rx_r \\ B_rGz \end{bmatrix} \tag{5}$$

This is compactly written as

$$\begin{bmatrix} \dot{x}_l \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_l & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x_l \\ x_r \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} x_l \\ x_r \end{bmatrix} \tag{6}$$

where

$$x_l = (x_c^T, z^T)^T \qquad A_l = \begin{bmatrix} A_c & B_cG \\ KC_c & A_{cz}+B_zG \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 0 \\ KC_r \end{bmatrix} \qquad \beta = [0 \quad B_rG]$$

$$A = \begin{bmatrix} A_l & 0 \\ 0 & A_r \end{bmatrix} \quad \text{and } x = \begin{bmatrix} x_l \\ x_r \end{bmatrix}$$

Assume that A_l and A_r are strictly Hurwitz matrices. That is, the controlled part is stable, and the residual modes have some damping.

Let (P,Q) be an ordered pair of positive definite symmetric $n\times n$ matrices $(n=n_c+n_r+p)$ satisfying

$$A^TP+PA=-Q \tag{7}$$

Let (P_l, Q_l) and (P_r, Q_r) be ordered pairs of $n_l\times n_l$ and $n_r\times n_r$ positive definite symmetric matrices $(n_l=n_c+p)$ satisfying

$$A_l^TP_l+P_lA_l=-Q_l \tag{8}$$

$$A_r^TP_r+P_rA_r=-Q_r \tag{9}$$

The spectral norm of a matrix H is defined as

$$\|H\|_s=(\text{maximum eigenalue of } H^TH)^{1/2}$$

$\|\cdot\|$ denotes the Euclidian norm of a vector.

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$\lambda_M(\ )$ and $\lambda_m(\ )$ denote maximum and minimum eigenvalues of a real symmetric matrix. $\triangleq$ denotes equality by definition.

Theorem I. System (6) is asymptotically stable if

$$\max(\|KC_r\|_s,\ \|B_rG\|_s) < \lambda_m(Q)/2\lambda_M(P) \triangleq \beta_1 \tag{10}$$

Proof: Consider the Lyapunov function

$$V(x) = x^{\mathrm{T}}Px$$
$$\dot{V}(x) = -x^{\mathrm{T}}Qx + 2x^{\mathrm{T}}P\Gamma x$$

where Q is a positive definite matrix, and

$$\Gamma = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$$

$$2x^{\mathrm{T}}P\Gamma x \leq 2\|Px\|\,\|\Gamma x\| \leq 2\|P\|_s\|\Gamma\|_s\|x\|^2$$

Therefore, V is negative definite if

$$\|\Gamma\|_s < \lambda_m(Q)/2\lambda_M(P) \tag{11}$$

But

$$\|\Gamma\|_s^2 = \lambda_M(\Gamma^{\mathrm{T}}\Gamma) = \lambda_M\left\{\begin{bmatrix} \beta^{\mathrm{T}}\beta & 0 \\ 0 & \alpha^{\mathrm{T}}\alpha \end{bmatrix}\right\}$$

Substituting for α and β from their definitions given earlier,

$$\|\Gamma\|_s^2 = \lambda_M\left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (B_rG)^{\mathrm{T}}(B_rG) & 0 & 0 \\ 0 & 0 & (KC_r)^{\mathrm{T}} & (KC_r) \end{bmatrix}\right\}$$

$$= \max\{\|B_rG\|_s^2, \|KC_r\|_s^2\}$$

$$\|\Gamma\|_s = \max\{\|B_rG\|_s, \|KC_r\|_s\}$$

Inequality (10) can be obtained by substituting the above expression for $\|\Gamma\|_s$ in Eq. (11).

Lemma: System (6) is asymptotically stable if there exist two positive scalars δ_1 and δ_2 such that

$$\delta_1\lambda_M(P_1)\|KC_r\|_s + \delta_2\lambda_M(P_r)\|B_rG\|_s < \sqrt{\delta_1\delta_2\lambda_m(Q_1)\lambda_m(Q_r)} \tag{12}$$

Proof: Equation (6) consists of two interconnected subsystems. Choose Lyapunov functions:

$$V_1(x_1) = x_1^{\mathrm{T}}P_1x_1 \quad \text{and} \quad V_2(x_r) = x_r^{\mathrm{T}}P_rx_r$$

where P_1 and P_r are as described earlier, and satisfy Eqs. (8) and (9).

Consider the uncoupled subsystem 1, which is given by

$$\dot{x}_1 = A_1x_1$$

The spectral norm of a positive definite symmetric matrix P_1 is also defined as

$$\|P_1\|_s = \max_{x_1}\frac{x_1^{\mathrm{T}}P_1x_1}{x_1^{\mathrm{T}}x_1}$$

Since P_1 is symmetric,

$$\|P_1\|_s = \lambda_M(P_1)$$

and

$$x_1^{\mathrm{T}}P_1x_1 \leq \lambda_M(P_1)\|x_1\|^2$$

Since P_1 is also positive definite, it can be proved that

$$\max\frac{x_1^{\mathrm{T}}x_1}{x_1^{\mathrm{T}}P_1x_1} = \frac{1}{\lambda_m(P_1)}$$

Therefore

$$\|x_1\|^2\lambda_m(P_1) \leq x_1^{\mathrm{T}}P_1x_1$$

Thus, for subsystem 1,

$$\lambda_m(P_1)\|x_1\|^2 \leq V_1(x_1) \leq \lambda_M(P_1)\|x_1\|^2$$

Considering the derivative, which is given by

$$\dot{V}_1(x_1) = -x_1^{\mathrm{T}}Q_1x_1$$

it can be shown in a similar manner that

$$\dot{V}_1(x_1) \leq -\lambda_m(Q_1)\|x_1\|^2$$

Also,

$$\left\|\frac{\partial V_1}{\partial x_1}\right\| \leq 2\lambda_M(P_1)\|x_1\|$$

Similar inequalities can be obtained for the residual subsystem. Theorem 2 of Ref. 4 can now be applied to obtain the sufficient condition, Eq. (12).

Theorem II.: System (6) is asymptotically stable if

$$\|KC_r\|_s\ \|B_rG\|_s < \frac{\lambda_m(Q_1)\lambda_m(Q_r)}{4\lambda_M(P_1)\lambda_M(P_r)} \triangleq \beta_2^2 \tag{13}$$

Proof: Scalars δ_1 and δ_2 in the lemma can be chosen to obtain the least conservative form of the sufficient condition Eq. (12). Letting $r = \sqrt{\delta_2/\delta_1}$, Eq. (12) can be expressed as

$$r^2\lambda_M(P_r)\|B_rG\|_s - r\sqrt{\lambda_m(Q_1)\lambda_m(Q_r)} + \lambda_M(P_1)\|KC_r\|_s < 0 \tag{14}$$

Minimization of the left-hand side of Eq. (14) with respect to r gives Eq. (13), which is the least conservative sufficient condition of the form of Eq. (12).

Remark 1. The lemma was obtained using a result in Ref. 4, which had used weighted sums of scalar Lyapunov functions of subsystems of the composite system. It has been shown in Ref. 4 that this approach generally yields less conservative and therefore better sufficient conditions for stability than the vector Lyapunov function approach of Bailey.[5]

Table 1 Bending mode frequencies and assumed damping ratios of example beam

Mode number i	Damping ratio ρ_i	Natural frequency ω_i, rad/s
1	0.00500	0.4275
2	0.00550	1.1718
3	0.00605	2.2968
4	0.00665	3.7967
5	0.00732	5.6717
6	0.00805	7.9215
7	0.00886	10.5460

Remark 2. Condition (13) in Theorem II can also be independently obtained using the results of Grujic and Siljak[6] for interconnected systems.

Let $V_l(x_l) = \sqrt{x_l^T P_l x_l}$ be a Lyapunov-type function for the uncoupled subsystem 1. It can be shown (in a way similar to the proof of the lemma) that

$$\sqrt{\lambda_m(P_l)}\,\|x_l\| \leq V_l(x_l) \leq \sqrt{\lambda_M(P_l)}\,\|x_l\|$$

$$\dot{V}_l(x_l) \leq -\tfrac{1}{2}\,\frac{\lambda_m(Q_l)}{\sqrt{\lambda_M(P_l)}}\,\|x_l\|$$

$$\left(\frac{\partial V_l}{\partial x_l}\right)^T \alpha x_r \leq \sqrt{\lambda_M(P_l)}\,\|\alpha\|_s \|x_r\|$$

Similar inequalities can be obtained for the residual subsystem assuming

$$V_2(x_r) = \sqrt{x_r^T P_r x_r}$$

Theorem 1 of Ref. 6 can now be applied to obtain Eq. (13).

Discussion and Numerical Results

Theorems I and II, proved above, give two different sufficient conditions for stability. However, the following drawbacks are normally associated with Lyapunov mehods: 1) the conditions obtained are conservative, and 2) considerable arbitrariness is involved in the selection of Lyapunov functions. The choice of matrices Q, Q_l and Q_r in Eqs. (7), (8), and (9) is arbitrary, the only constraint being that they are positive definite. More investigation is needed in the selection of these matrices in order to obtain the least conservative bounds.

A rigid-body-plus-seven-mode, normal coordinate, planar model of a uniform free-free beam was chosen for evaluating the stability bounds (for a development of the equations for a uniform free-free beam, see Ref. 7). Bending mode frequencies and assumed damping ratios of the beam are given in Table 1. One torque actuator, one attitude sensor and one rate sensor were used. Bounds β_l and β_2 were computed with matrices Q, Q_l and Q_r equal to identity matrices of appropriate dimensions, for a nominal set of LQG regulator and estimator gains (G and K). The regulator was designed to control rigid-body-plus-first two modes, and the estimator was designed to estimate rigid-body-plus-first five modes. For this case, β_l was 0.184×10^{-7}, and β_2 was 0.553×10^{-5}. Thus, bound β_2 is far less conservative than bound β_l (by a factor of about 300). In fact, if Q is block-diagonal, the l-system and r-system give uncoupled Lyapunov matrix equations. In this case, it can be seen from Eqs. (10) and (13), that $\beta_l \leq \beta_2$. In addition, Eq. (13) is consistent with the fact that the absence of either spillover term assures stability. Thus, Eq. (13) appears to be a better sufficient condition.

Conclusions

Two sufficient conditions were derived via Lyapunov methods for asymptotic stability of large space structures using a class of reduced-order controllers. These conditions give allowable bounds on the spectral norms of control and observation "spillover" terms. The sufficient condition given in Eqs. (13) appears to be less conservative, and should be useful as a design tool for the control of large space structures.

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The Orientation Vector Differential Equation

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Introduction

THE orientation or Euler vector $\psi(t)$ relates a rotating body frame to a reference frame at any time t, in that rotating a reference frame about an axis coincident with $\psi(t)$ through an angle which is equal to the magnitude of $\psi(t)$ will bring the reference frame into coincidence with the body frame. Thus, if we define θ as the magnitude of the rotation and $\boldsymbol{u}$ as the unit vector about which the rotation has occurred, then $\psi(t) = \theta(t)\boldsymbol{u}(t)$.

In Ref. 1 Bortz derived an expression for the time rate of change of the orientation vector and discussed some of its applications. The vector $\dot{\psi}(t)$ was shown to have two components: $\omega(t)$, the component due to inertially measurable angular motion (angular velocity vector), and $\dot{\sigma}(t)$, the component due to noninertially measurable angular motion (noncommutativity rate vector), where $\psi(t)$ and $\dot{\sigma}(t)$ are orthogonal. The latter component reflects the fact that the orientation of a rigid body after a sequence of rotations depends on the order of the rotations as well as the magnitude, and axis orientation of the individual rotations.

While Bortz's derivation is on the whole straightforward, the derivation is also rather lengthy and involves a number of algebraic manipulations. In this paper we give a derivation of $\dot{\psi}(t)$ which we believe to be much simpler. Most of the simplification comes from exploiting a kinematical property of the Euler-Rodrigues rotation parameters. This approach was not used in Ref. 1. The three-parameter Euler-Rodrigues method is a special case of the more widely known four-parameter Euler method.[2]

Derivation

Before proving the main result, we first define some terms and then state two lemmas which will help expedite the derivation of $\dot{\psi}(t)$.

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